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ON AUTOMORPHIC BANACH SPACES

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ABSTRACT

A Banach space X will be called **extensible** if every operator $E \to X$ from a subspace $E \subset X$ can be extended to an operator $X \to X$. Denote by dens X. The smallest cardinal of a subset of X whose linear span is dense in X, the space X will be called **automorphic** when for every subspace $E \subset X$ every into isomorphism $T : E \to X$ for which $\operatorname{dens} X/E = \operatorname{dens} X/TE$ can be extended to an automorphism $X \to X$. Lindenstrauss and Rosenthal proved that c_0 is automorphic and conjectured that c_0 and ℓ_2 are the only separable automorphic spaces. Moreover, they ask about the extensible or automorphic character of $c_0(\Gamma)$, for Γ uncountable. That $c_0(\Gamma)$ is extensible was proved by Johnson and Zippin, and we prove here that it is automorphic and that, moreover, every automorphic space is extensible while the converse fails. We then study the local structure of extensible spaces, showing in particular that an infinite dimensional extensible space cannot contain uniformly complemented copies of ℓ_p^n, $1 \le p < \infty$, $p \ne 2$. We derive that infinite dimensional spaces such as $L_p(\mu)$, $p \ne 2$, $C(K)$ spaces not isomorphic to c_0 for K metric compact, subspaces of c_0 which are not isomorphic to c_0, the Gurarij space, Tsirelson spaces or the Argyros-Deliyanni HI space cannot be automorphic.

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1. Introduction

In [13] it was proved that:

THEOREM 1.1 (Lindenstrauss-Rosenthal): *Let E be a subspace of c_0 and let $T : E \to c_0$ be an into isomorphism such that* $\dim c_0/E = \dim c_0/TE = \infty$. *There is an automorphism $\widehat{T}$ of c_0 such that $\widehat{T}|_E = T$.*

Let us call a Banach space **automorphic** if for every closed subspace $E \subset X$ and every into isomorphism $T : E \to X$ with $\operatorname{dens}(X/E) = \operatorname{dens}(X/TE)$ there is an automorphism $\widehat{T}$ of X such that $\widehat{T}|_E = T$.

Let us notice that the condition on the quotient spaces is necessary for the existence of the automorphism $\widehat{T}$ of X since it naturally induces an isomorphism between the quotient spaces X/E and X/TE. We remark that the definition of the **density character** we handle is slightly unusual: for a Banach space X, by $\operatorname{dens} X$ we mean the smallest cardinal of a subset of X whose linear span is dense in X. This definition coincides with the standard one when X is infinite dimensional and is more adequate to the problems considered here since it coincides with the dimension when the space is finite dimensional.

In [13] the following still unsolved conjecture is formulated: The Banach spaces c_0 and ℓ_2 are the only separable infinite dimensional automorphic spaces. This paper is mostly devoted to the study of aspects of the Lindenstrauss-Rosenthal conjecture related to the local structure of automorphic spaces as well as their properties regarding the extension of operators. In the literature there are several examples of "partially automorphic" spaces: for instance, it is known that for every subspace $E \subset \ell_1$ and every into isomorphism $T : E \to \ell_1$ such that ℓ_1/E and ℓ_1/TE are infinite dimensional $\mathcal{L}_1$-spaces there is an automorphism $\widehat{T}$ of ℓ_1 extending T (see [9] or [5]); and the space $C[0,1]$ has the analogous property for subspaces of c_0 (see [12] or [5]). A homological approach to the study of automorphic and partially automorphic spaces appears in [5], including a unified method of proof for all the previous results.

Apart from c_0 and the trivially automorphic spaces $\ell_2(\Gamma)$, no other automorphic space has been discovered so far. From [13] we know that ℓ_∞ is pretty close to being automorphic although it is not: the automorphism could not exist when the quotient spaces ℓ_∞/E and ℓ_∞/TE are both infinite dimensional separable reflexive spaces. The space $c_0(\Gamma)$, with Γ a non-countable set, seems to be a natural candidate to be automorphic. Lindenstrauss and Rosenthal ask in [13] whether that is true and Zippin suggests in [26, Remark 5.8] that $c_0(\Gamma)$ is likely to be automorphic. The main theorem in Section 2 (Theorem 2.1) shows that $c_0(\Gamma)$ is indeed an automorphic space. In Section 3 we carry out an approach to the conjecture that c_0 and ℓ_2 are the only separable infinite dimensional automorphic spaces by studying *extension of operator properties* of automorphic spaces. As we will show, a necessary condition for a Banach space to be automorphic is to be **extensible** in the following sense

Definition 1.1 (Extensible space): An infinite dimensional Banach space X is said to be extensible if for every subspace $E \subset X$, every operator $T : E \to X$ can be extended to an operator on X.

Although c_0 and ℓ_2 are extensible, not every extensible space is automorphic, as ℓ_∞ shows. On the other hand, we will show that any automorphic space is extensible. Hence, $c_0(\Gamma)$ is also extensible; this was asked by Lindenstrauss and Rosenthal in [13] and answered by Johnson and Zippin in [8]. In Section 4, we study the local structure of extensible spaces. For this purpose we introduce and study the notion of **uniformly finitely extensible pair** of Banach spaces. Denote $\lambda(E, X)$ the **relative projection constant** of a subspace E of X. We prove that an extensible space cannot contain sequences of uniformly pairwise isomorphic finite dimensional subspaces $E_n \subset X$ and $F_n \subset X$ such that $\sup_n \lambda(E_n, X) < \infty$ and $\lim_n \lambda(F_n, X) = +\infty$. In particular, if X contains uniformly complemented copies of ℓ_p^n, $1 \leq p < \infty$, $p \neq 2$; then X cannot be extensible. We then present a list of Banach spaces which cannot be extensible (hence they cannot be automorphic); among those, and contrary to Zippin's expectations (see [26, p. 1729]), hereditarily indecomposable spaces need not be automorphic. The list of non-automorphic spaces includes $L_p(\mu)$ spaces, $p \neq 2$, which answers a question of Lindenstrauss and Rosenthal in [13]; all subspaces of c_0 other than c_0 itself; Tsirelson spaces and the Gurarij space, against Tokarev's expectations in [23]. The local approach to the Lindenstrauss-Rosenthal conjecture in Section 4 suggests that automorphic spaces should be locally similar to either ℓ_2, c_0 or to spaces with "badly-normed" finite rank projections. According to this remark, we formulate in Section 5 some open questions.

By **operator** we mean bounded linear operator, by **space** we mean Banach space. An **into isomorphism** is an injective operator $T : Y \to X$ with closed image. We shall use the notation $X \sim Y$ to mean that X and Y are isomorphic spaces.

2. $c_0(\Gamma)$ is automorphic

Let Γ be any set. We shall use for a subset $X \subset c_0(\Gamma)$ the notation

$$\operatorname{supp} X = \{\gamma \in \Gamma : x(\gamma) \neq 0 \text{ for some } x \in X\}.$$

For a subset $\Delta \subset \Gamma$, let $\neg\Delta = \Gamma\backslash\Delta$; the space $c_0(\Delta)$ denotes the set of elements of $c_0(\Gamma)$ having support in Δ; by P_Δ we denote the natural projection of $c_0(\Gamma)$ onto $c_0(\Delta)$ and for a subspace $X \subset c_0(\Gamma)$, $X_\Delta = X \cap c_0(\Delta)$. Let A be an arbitrary set of indices. For a collection (X_α), $\alpha \in A$, of (linearly independent) closed subspaces of a closed subspace $X \subset c_0(\Gamma)$ one has $X = c_0(X_\alpha, A)$ when the closed linear span of $\bigcup_\alpha X_\alpha$ is equal to X and for any $\alpha \neq \beta$

$$\operatorname{supp} X_\alpha \cap \operatorname{supp} X_\beta = \emptyset.$$

THEOREM 2.1: *Let Γ be an uncountable set. The space $c_0(\Gamma)$ is automorphic.*

Indeed, in [26, Remark 5.8] Zippin pointed to the following decomposition lemma [8, Lemma 2] as an instrument for proving Theorem 2.1 (also, Castillo and Johnson suggested that possibility in private communications).

LEMMA 2.2 (Johnson-Zippin): *Let Γ be an uncountable set and let X be a closed subspace of $c_0(\Gamma)$. For every countable subset $\Gamma' \subset \Gamma$ there is a countable subset $\Gamma' \subset \Delta \subset \Gamma$ such that $P_\Delta x \in X$ for every $x \in X$.*

We shall use for the proof of Theorem 2.1 a refined versions of this lemma. For the rest of this section, X and Y will be isomorphic infinite dimensional closed subspaces of $c_0(\Gamma)$ such that $c_0(\Gamma)/X$ and $c_0(\Gamma)/Y$ are infinite dimensional spaces and $T : X \to Y$ will be an isomorphism. Let us notice that if $\operatorname{dens} X < \operatorname{card}\Gamma$ then there is a subset $\Delta \subset \Gamma$ such that $X \cup Y \subset c_0(\Delta)$ and $\operatorname{card}\Delta = \operatorname{dens} X$. Therefore, one can assume that $\operatorname{dens} X = \operatorname{card}\Gamma$ and $\operatorname{supp}(X \cup Y) = \Gamma$.

LEMMA 2.3: *For every countable subset $\Gamma' \subset \Gamma$ there is a countable subset $\Gamma' \subset \Delta \subset \Gamma$ such that*

(1) *$P_\Delta X \subset X$ and $P_\Delta Y \subset Y$;*
(2) *$P_{\neg\Delta} X \subset X$ and $P_{\neg\Delta} Y \subset Y$.*

Proof. Let us construct a sequence (Δ_n) of countable sets as follows: Let $\Delta_1 \supset \Gamma'$ be a countable set for which $P_{\Delta_1} x \in X$ for every $x \in X$ (Lemma 2.2). Let $\Delta_2 \supset \Delta_1$ be a countable set such that $P_{\Delta_2} y \in Y$ for every $y \in Y$. Let $\Delta_3 \supset \Delta_2$ be such that $P_{\Delta_3} x \in X$ for every $x \in X$. We obtain in this way an increasing sequence (Δ_n) of countable sets such that for every n, $P_{\Delta_{2n-1}}X \subset X$ and $P_{\Delta_{2n}}Y \subset Y$. The set $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$ has the properties (1) and (2). ∎

Lemma 2.4: *For every countable set $\Gamma' \subset \Gamma$ there is a countable set $\Gamma' \subset \Delta \subset \Gamma$ such that* (1) *and* (2) *hold and $TX_\Delta = Y_\Delta$ and $TX_{\neg\Delta} = Y_{\neg\Delta}$.*

Proof. Let us construct an increasing sequence (Δ_n) of countable sets as follows: Take a countable set $\Delta_1 \supset \Gamma'$ for which (1) and (2) hold and $TX_{\Gamma'} \subset Y_{\Delta_1}$. Take $\Delta_2 \supset \Delta_1$ for which (1) and (2) hold and $T^{-1}Y_{\Delta_1} \subset X_{\Delta_2}$. Take $\Delta_3 \supset \Delta_2$ such that (1) and (2) hold and $TX_{\Delta_2} \subset Y_{\Delta_3}$. Take $\Delta_4 \supset \Delta_3$ for which (1) and (2) hold and $T^{-1}Y_{\Delta_3} \subset X_{\Delta_4}$ and so on. The set $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$ verifies conditions (1) and (2). Obviously $TX_\Delta \subset Y_\Delta$ and $TX_{\neg\Delta} \subset Y_{\neg\Delta}$. Let us see that $TX_\Delta = Y_\Delta$: let $y \in Y_\Delta$, then $y = Tx$ for some $x = x_1 + x_2$ with $x_1 \in X_\Delta$ and $x_2 \in X_{\neg\Delta}$. So, $y = Tx_1 + Tx_2$ and $Tx_2 \in Y_\Delta \subset P_\Delta(c_0(\Gamma))$. But $Tx_2 \in Y_{\neg\Delta} \subset P_{\neg\Delta}(c_0(\Gamma))$, therefore $Tx_2 = 0$ and $y = Tx_1 \in TX_\Delta$. The equality $TX_{\neg\Delta} = Y_{\neg\Delta}$ can be checked in the same way. ∎

Lemma 2.5: *Let* $\operatorname{dens} c_0(\Gamma)/X = \operatorname{dens} c_0(\Gamma)/Y$. *There is a set A of cardinality* $\operatorname{card}\Gamma$ *such that the set Γ admits a decomposition $\Gamma = \bigcup_{\alpha\in A} \Delta_\alpha$ in pairwise disjoint countable sets such that for every $\alpha \in A$:*

a) $TX_{\Delta_\alpha} = Y_{\Delta_\alpha}$;
b) $X = c_0(X_{\Delta_\alpha}, A)$ *and* $Y = c_0(Y_{\Delta_\alpha}, A)$.

Proof. In the set of families $F_A = (\Delta_\alpha)_{\alpha\in A}$ of pairwise disjoint subsets of Γ verifying the four conditions of Lemma 2.4, for which A is a set of cardinality $\operatorname{card}\Gamma$ we introduce the natural order: $(\Delta_\alpha)_{\alpha\in A} \le (\Delta'_\alpha)_{\alpha\in A'}$ if and only if $A \subset A'$ and for all $\alpha \in A$ one has $\Delta_\alpha = \Delta'_\alpha$. This is an inductive order, since if (F_{A_j}) is a chain, then one can set as its upper bound the family $F_{\bigcup A_j}$: whenever $\alpha \in A_j$ the set Δ_α is uniquely defined. That $\operatorname{card} \bigcup_j A_j = \operatorname{card}\Gamma$ is guaranteed by $\operatorname{card} \bigcup_j A_j = \operatorname{card} \bigcup_j \bigcup_{\alpha\in A_j} \Delta_\alpha \le \operatorname{card}\Gamma$. Therefore, there must be a maximal family $(\Delta^m_\alpha)_{\alpha\in A}$. If $\Gamma \setminus \bigcup_{\alpha\in A} \Delta^m_\alpha$ is empty, we are done; if it is countable, Lemma 2.4 and the maximality of the family yield a contradiction. If it is finite then add this finite set of points to some set Δ^m_α. ∎

Lemma 2.6: *Let A be an infinite index set. Let $\{B_\alpha\}_{\alpha\in A}$ and $\{C_\alpha\}_{\alpha\in A}$ be families of pairwise disjoint sets, each of them countable (finite or infinite) or empty, in such a way that $\bigcup_{\alpha\in A} B_\alpha$ and $\bigcup_{\alpha\in A} C_\alpha$ have the same infinite cardinal. Then A can be decomposed into a disjoint union of countable sets,*

$A = \bigcup_{i\in I} A_i$, *such that for any* i *either both* $\bigcup_{\alpha\in A_i} B_\alpha$ *and* $\bigcup_{\alpha\in A_i} C_\alpha$ *are infinite or both are empty.*

Proof. We set $A_0^0 = \{\alpha \in A : B_\alpha = \emptyset = C_\alpha\}$; $A_1^0 = \{\alpha : B_\alpha \neq \emptyset = C_\alpha\}$; $A_0^1 = \{\alpha : B_\alpha \emptyset \neq C_\alpha\}$ and $A_1^1 = \{\alpha : B_\alpha \neq \emptyset \neq C_\alpha\}$. Assume that A_1^1 is infinite and set a decomposition $A_1^1 = \bigcup_{i\in I} A_i$ into infinite countable sets. If $\mathrm{card} A_1^0 > \mathrm{card} A_1^1$ then the hypothesis forces $\mathrm{card} A_1^0 = \mathrm{card} A_0^1$ and let H be a bijection between those sets. If $A_1^0 = \bigcup_{j\in J} A_j$ and $A_0^0 = \bigcup_{k\in K} A_k$ are decomposition into infinite countable sets then $A = \bigcup_i A_i \cup \bigcup_j (A_j \cup H(A_j)) \cup \bigcup_k A_k$ is the desired decomposition. If, however, there exist injections $F : A_1^0 \to A_1^1$ and $G : A_0^1 \to A_1^1$ then the decomposition is $\bigcup_i (A_i \cup F^{-1}(A_i) \cup G^{-1}(A_i)) \cup \bigcup_k A_k$. If A_1^1 is finite and both A_1^0 and A_0^1 are finite, then the decomposition is $A = \bigcup_{k\in K} A_k \cup (A_1^1 \cup A_0^1 \cup A_1^0)$. If, say, A_0^1 is finite and A_1^0 is infinite, then the same decomposition of A works since now A_1^0 must be countable. Finally, if A_1^0 and A_0^1 are both infinite, then $\mathrm{card} A_1^0 = \mathrm{card} A_0^1$; fix a bijection $H : A_1^0 \to A_0^1$ and an injection $F : A_1^1 \to A_1^0$ and the decomposition is $A = \bigcup_j (A_j \cup H(A_j) \cup F^{-1}(A_j)) \cup \bigcup_k A_k$. ∎

Lemma 2.7: *Let* $\mathrm{dens}\, c_0(\Gamma)/X = \mathrm{dens}\, c_0(\Gamma)/Y$ *be infinite. The set* Γ *admits a decomposition* $\Gamma = \bigcup_{i\in I} \Gamma_i$ *into pairwise disjoint countable sets such that for every* i*:*

a) $TX_i = Y_i$ *where* $X_i := X_{\Gamma_i}$, $Y_i := Y_{\Gamma_i}$;
b) *either both* $c_0(\Gamma_i)/X_i$ *and* $c_0(\Gamma_i)/Y_i$ *are infinite dimensional or both are equal to* 0;
c) $X = c_0(X_i, I)$ *and* $Y = c_0(Y_i, I)$.

Proof. Let us take a set A of cardinality equal to Γ and take a decomposition $\Gamma = \bigcup_{\alpha\in A} \Delta_\alpha$ like in Lemma 2.5. For every $\alpha \in A$, the embedding $X_{\Delta_\alpha} \hookrightarrow c_0(\Delta_\alpha)$ induces a separable quotient $c_0(\Delta_\alpha)/X_{\Delta_\alpha}$ (it could be zero). Consider for every $\alpha \in A$ sets B_α and C_α of cardinality equal to $\mathrm{dens}\, c_0(\Delta_\alpha)/X_{\Delta_\alpha}$ and $\mathrm{dens}\, c_0(\Delta_\alpha)/Y_{\Delta_\alpha}$, respectively, in such a way that the families $(B_\alpha)_{\alpha\in A}$ and $(C_\alpha)_{\alpha\in A}$ are formed by pairwise disjoint elements. By the hypothesis,

$$\mathrm{card} \bigcup_{\alpha\in A} B_\alpha = \mathrm{dens}\, (c_0(\Gamma)/X) = \mathrm{dens}\, (c_0(\Gamma)/Y) = \mathrm{card} \bigcup_{\alpha\in A} C_\alpha$$

and it is infinite. We can apply Lemma 2.6 to obtain a decomposition $A = \bigcup_{i\in I} A_i$ such that for each $i \in I$ if we set $\Gamma_i = \bigcup_{\alpha\in A_i} \Delta_\alpha \subset \Gamma$, then the decomposition $\Gamma = \bigcup_{i\in I}\Gamma_i$ verifies b):

$$\text{dens } c_0(\Gamma_i)/X_i = \text{card} \bigcup_{\alpha\in A_i} B_\alpha = \text{card} \bigcup_{\alpha\in A_i} C_\alpha = \text{dens } c_0(\Gamma_i)/Y_i.$$

That a) and c) also hold follows directly from Lemma 2.5 and from the observations $X_i = c_0(X_{\Delta_\alpha}, A_i)$ and $Y_i = c_0(Y_{\Delta_\alpha}, A_i)$. ∎

Proof of Theorem 2.1. If both quotients are finite dimensional, our definition of the density character implies their dimensions coincide and therefore the existence of the automorphism is clear. Otherwise, consider a partition (Γ_i) of Γ as in Lemma 2.7. Let us extend every $T|_{X_i}$ to an automorphism $\widehat{T}_i$ of $c_0(\Gamma_i)$ using the Lindenstrauss-Rosenthal theorem. Moreover, from the proof of Theorem 1 in [13] one can check that there is a function $g : [1,\infty) \to [1,\infty)$ such that for some constant C and for every i, $\|\widehat{T}_i\|\|\widehat{T}_i^{-1}\| \le g(\|T|_{X_i}\|\|(T|_{X_i})^{-1}\|) \le Cg(\|T\|\|T^{-1}\|)$. The final extension is obvious: for every $x = \sum_i x_i$, $x_i \in X_i$, put $\widehat{T}x = \sum_i \widehat{T}_i x_i$. ∎

3. Extensible spaces

As we mentioned in the introduction, the spaces ℓ_1, $C[0,1]$ and ℓ_∞ have a *partially* automorphic character. If X is one of those spaces, or c_0 or ℓ_2, there is a class $\mathcal{T}_X$ of injective isomorphisms into X such that for every $T \in \mathcal{T}_X$ there exists an automorphism $\widehat{T}$ of X extending T. Let us observe that in all the previous cases not only the into isomorphisms of the class $\mathcal{T}_X$ can be extended to X, but even every operator $S : E \to X$ defined on a subspace E of X for which the embedding $E \to X$ is in $\mathcal{T}_X$ can also be extended to X: by Lindenstrauss' theorem in [9], given a subspace $E \subset \ell_1$, every operator $E \to \ell_1$ extends to ℓ_1 when ℓ_1/E is a $\mathcal{L}_1$-space; by a combination of Lindenstrauss-Pełczyński theorem [12] and Sobczyk's theorem [24] if E is a subspace of $C[0,1]$ which embeds into c_0, the operators $E \to C[0,1]$ extend to $C[0,1]$; and, as is well-known, all operators into ℓ_∞ extend to any superspace. The two already known automorphic spaces also enjoy the corresponding extension of operators property: every operator on a subspace of $\ell_2(I)$ extends to $\ell_2(I)$ and, finally, operators $E \to c_0$ extend to c_0. After these remarks the introduction of the notion of extensible space (Definition 1.1) is meaningful.

Let us prove that every automorphic space is extensible. For $\varepsilon > 0$, we call an operator $T : X \to Y$ an **ε-isometry** provided for every $x \in X$, $(1-\varepsilon)\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\|$.

PROPOSITION 3.1: *Let E be a closed subspace of a Banach space X and let $\varepsilon > 0$. If every ε-isometry $E \to X$ can be extended to an operator $X \to X$, then every operator $E \to X$ can be extended to an operator $X \to X$.*

Proof. Assume there is an operator $T : E \to X$, $\|T\| = 1$, which cannot be extended to an operator $X \to X$. For every operator $S : X \to X$ and for every $\varepsilon > 0$ the operator $S|_E + \varepsilon T$ cannot be extended either. In particular, if we take S to be the identity, there is an ε-isometry $E \to X$ which does not extend to X. ∎

THEOREM 3.2: *Every automorphic space X is extensible.*

Proof. Assume there is a subspace $E \subset X$ and an operator $T : E \to X$, $\|T\| = 1$, which cannot be extended to X, then X/E is infinite dimensional. Then, by Proposition 3.1, for some $\varepsilon > 0$ the operator $R = id|_E + \varepsilon T$ is an ε-isometry which cannot be extended to X. Let us show that $\mathrm{dens} X/E = \mathrm{dens} X/RE$. Let $\delta > 0$ and let $\{x_i : i \in I\}$ be a set in the unit sphere of X such that $\mathrm{card} I = \mathrm{dens} X/E$ and for every $i, j \in I$,

$$\inf\{\|x_i - x_j + y\| : y \in E\} > 1 - \delta.$$

Then, because the subspaces E and RE are very close if ε is sufficiently small,

$$\inf\{\|x_i - x_j + y\| : y \in RE\} > 1 - \varphi(\varepsilon, \delta),$$

where $\varphi(\varepsilon, \delta)$ is very small if ε and δ are sufficiently small. Hence, X/RE is also infinite dimensional and $\mathrm{dens} X/RE \ge \mathrm{dens} X/E$. Taking R^{-1} instead of R, we obtain $\mathrm{dens} X/RE \le \mathrm{dens} X/E$. So, $\mathrm{dens} X/RE = \mathrm{dens} X/E$ and X is not automorphic. ∎

As a corollary of Theorems 2.1 and 3.2 one obtains the result of [8]:

COROLLARY 3.3: *Let Γ be an uncountable set. The space $c_0(\Gamma)$ is extensible.*

QUESTION 1: Is every separable extensible space automorphic?

Simple examples of non-extensible spaces are given by spaces containing pairs of isomorphic subspaces one of them complemented and the other uncomplemented. The space ℓ_1, which admits an uncomplemented copy of itself according to Bourgain's construction in [4], is a natural example. Also, separable $C(K)$ spaces non-isomorphic to c_0 cannot be extensible; indeed, it follows from a result of Amir [1] and Pełczyński [20, Section 9], that those spaces contain subspaces E and F with E complemented, F uncomplemented and $E \sim F \sim C(\omega^\omega)$.

4. A local approach to extensible spaces

Let us begin this section with more general and more detailed definitions which are in the spirit of [26, Definition 1.9] or those in [7].

Definition 4.1 (Extensible couple): Given a couple (X, Y) of Banach spaces and $\lambda \geq 1$ we will say that it is extensible (resp. compactly extensible) if every operator (resp., compact operator) $T : E \to Y$ from a subspace $E \subset X$ can be extended to an operator $\widehat{T} : X \to Y$. If the extension of the operator $\widehat{T}$ verifies $\|\widehat{T}\| \leq \lambda\|T\|$, we say that the couple (X, Y) is λ-extensible (resp., compactly λ-extensible). The couple (X, Y) is said to be finitely λ-extensible if every operator $T : E \to Y$ from a finite dimensional subspace $E \subset X$ can be extended to an operator $\widehat{T} : X \to Y$ with $\|\widehat{T}\| \leq \lambda\|T\|$. The couple (X, Y) will be called uniformly extensible (resp., uniformly compactly extensible, uniformly finitely extensible) if it is λ-extensible (resp., compactly λ-extensible, finitely λ-extensible) for some λ.

We give two examples: the Lindenstrauss-Pełczyński theorem [12] asserts that for every $C(K)$-space the couple $(c_0, C(K))$ is $(1+\varepsilon)$-extensible for every $\varepsilon > 0$; while Lindenstrauss' results in [10] mean that for every separable Banach space X and every $\mathcal{L}_{\infty,\lambda}$-space Y the couple (X, Y) is compactly $(\lambda+\varepsilon)$-extensible for every $\varepsilon > 0$. It is clear that X is extensible if and only if the couple (X, X) is extensible. The meaning of λ-extensible or compactly λ-extensible space should also be clear. Let us establish some connections between these notions. Note that if F is a finite codimensional closed subspace of a Banach space X, then the couple (X, Y) is uniformly finitely extensible if and only if the couple (F, Y) is uniformly finitely extensible. The proof of the following lemma is rather standard.

Lemma 4.1: *Assume that (X, Y) is not a uniformly finitely extensible couple. Then there are subspaces $E_n \subset X$ which form a finite dimensional Schauder decomposition of its closed linear hull E and operators $T_n : E_n \to Y$, $\|T_n\| = 1$, such that the norm of every extension of T_n onto X is not smaller than 2^{2n}.*

Proof. Let $\varepsilon > 0$. By hypothesis, there is a finite dimensional subspace $E_1 \subset X$ and a norm one operator $T_1 : E_1 \to Y$ such that the norm of every extension of T_1 to X is greater than or equal to 2^2. Let Φ_1 be a finite subset of the unit sphere $S(X^*)$ which $(1-\varepsilon)$-norms E_1, and let $\Phi_1^\top \subset X$ be its (finite codimensional) annihilator. Then the couple $(\Phi_1^\top, Y)$ is not uniformly finitely extensible. So, there is a finite dimensional subspace $E_2 \subset \Phi_1^\top$ and a norm one operator $T_2 : E_2 \to Y$ such that the norm of every extension of T_2 onto $\Phi_1^\top$ is greater than or equal to 2^4. Let Φ_2 be a finite subset of the sphere $S(X^*)$ which $(1-\varepsilon)$-norms $E_1 + E_2$, and let $\Phi_2^\top \subset X$ be its (finite codimensional) annihilator. The way of continuing the construction is clear. The conditions that Φ_n $(1-\varepsilon)$-norms $\sum_1^n E_i$ and $E_i \subset \Phi_n^\top$ for $i > n$, guarantee that (E_n) forms a finite dimensional Schauder decomposition of E. ∎

Proposition 4.2: *If the couple (X, Y) is compactly extensible, then it is uniformly finitely extensible. If the couple (X, Y) is finitely λ-extensible and Y is β-complemented in its bidual, then (X, Y) is $\lambda\beta$-extensible.*

Proof. To prove the first assertion, assume that (X, Y) is not uniformly finitely extensible. Let (E_n) be a finite dimensional decomposition for its closed linear hull E and let (T_n) be operators as in Lemma 4.1. We define the operator $T : E \to Y$ by

$$T\left(\sum\nolimits_1^\infty x_n\right) = \sum\nolimits_1^\infty 2^{-n} T_n x_n \ ,$$

where $x_n \in E_n$ and the series converges. The operator T is compact, and no extension $\hat{T} : X \to Y$ is possible.

To prove the second assertion, let $E \subset X$ be a subspace and let $T : E \to Y$ be an operator. Consider for each finite dimensional subspace $j_\alpha : E_\alpha \to E$ of E an extension $T_\alpha : X \to Y$ of $T j_\alpha$ having norm $\|T_\alpha\| \le \lambda \|T j_\alpha\|$. If $\mathcal{U}$ denotes a free ultrafilter on the set of finite dimensional subspaces of E that refines the Fréchet filter corresponding to the natural order, define the operator $T^{\mathcal{U}} : X \to Y^{**}$ by $T^{\mathcal{U}} x = w^* - \lim_{\mathcal{U}} T_\alpha x$ (by the Banach-Alaoglu theorem, this limit exists). If $P : Y^{**} \to Y$ is a projection with norm at most β, then $PT^{\mathcal{U}}$ extends T and verifies $\|PT^{\mathcal{U}}\| \le \beta\lambda$. ∎

It would be interesting to know if extensible implies uniformly extensible. The second part of Proposition 4.2 says that for reflexive Y extensibility, compact extensibility and uniform finite extensibility coincide. It would also be interesting to determine if compactly extensible and uniformly finitely extensible coincide at least in spaces with the approximation property.

Let us consider the connection between uniform finite extensibility and the local structure of a Banach space. By $\lambda(E,X)$ we denote the **relative projection constant** of a subspace E of X, which is defined as follows:

$$\lambda(E,X) = \inf\{\|P\| : P \text{ is a projection of } X \text{ onto } E\}.$$

When E is not complemented we set $\lambda(E,X) = \infty$. Relative projection constants are connected with the extension of isomorphisms as follows.

Lemma 4.3: *Let $E \subset X$, $F \subset Y$ be closed subspaces and let $T : E \to F$ be an isomorphism. Then for every extension $\hat{T} : X \to Y$ of T*

$$\|\hat{T}\| \geq \frac{\lambda(E,X)}{\lambda(F,Y)} \frac{1}{\|T^{-1}\|} \; . \tag{1}$$

Proof. Let $\varepsilon > 0$ and let P be a projection of Y onto F with $\|P\| < \lambda(F,Y)+\varepsilon$. Then the operator $Q = T^{-1}P\hat{T}$ is a projection of X onto E and

$$\|Q\| \leq \|T^{-1}\|(\lambda(F,Y)+\varepsilon)\|\hat{T}\| \; .$$

Since ε is arbitrary, the inequality (1) is clear. ∎

Let $\mathrm{d}(E,F)$ denote the Banach-Mazur distance between two (isomorphic) Banach spaces E and F. As an immediate consequence of Lemma 4.3 we obtain

Theorem 4.4: *Assume that the spaces X and Y contain sequences of finite dimensional subspaces E_n and F_n, respectively, with $\dim E_n = \dim F_n$, and such that*

$$\frac{\lambda(E_n,X)}{\lambda(F_n,Y)\mathrm{d}(E_n,F_n)} \to \infty \quad \text{as} \quad n \to \infty. \tag{2}$$

Then the couple (X,Y) is not uniformly finitely extensible. In particular, if $X = Y$, then the space X is not extensible and therefore, it cannot be automorphic.

Proof. Let $\varepsilon > 0$ and, for every n, let $T_n : E_n \to F_n$ be an isomorphism with $\|T_n\| = 1$ and $\|T_n^{-1}\| < \mathrm{d}(E_n,F_n)+\varepsilon$. If $\hat{T}_n$ denotes any extension of T_n, by (2) and Lemma 4.3 one necessarily has $\lim_{n\to\infty} \|\hat{T}_n\| = \infty$. ∎

Maurey's extension theorem in [16] ensures that for every space X of type 2, every space Y of cotype 2, the couple (X, Y) is uniformly extensible. On the other hand, under some additional assumptions (see [18], [6]) if X does not have type 2, there is an operator from some subspace of X into ℓ_2 which cannot be extended to X. The next corollary of Theorem 4.4 is close to this result.

Corollary 4.5: *Let X contain a sequence of subspaces E_n uniformly isomorphic to ℓ_2^n such that $\lambda(E_n, X) \to \infty$. Let Y contain a sequence of subspaces which are uniformly complemented and uniformly isomorphic to ℓ_2^n. Then the couple (X, Y) is not uniformly finitely extensible.*

Corollary 4.5 applies that when $p = \sup\{p' : X \text{ is a space of type } p'\} < 2$ (by [17], such space contains uniformly ℓ_p^n, $p < 2$, and by [3], a space with this property contains a sequence of uniformly Euclidean subspaces E_n such that $\lambda(E_n, \ell_p^n) \to \infty$) and Y is B-convex (by [21], such space contains a sequence of subspaces which are uniformly complemented and uniformly isomorphic to ℓ_2^n). It is well-known that for every $1 \leq p < \infty$, $p \neq 2$, there is a sequence of subspaces $E_n \subset \ell_p^n$, uniformly isomorphic to $\ell_p^{k(n)}$ such that $\lambda(E_n, \ell_p^n) \to \infty$ as $n \to \infty$. This was proved: for $2 < p < \infty$ and $1 < p < 4/3$ in [22]; for $1 < p < 2$ in [3] and for $p = 1$ in [4]. Therefore we have

Corollary 4.6: *Let $1 \leq p < \infty$, $p \neq 2$, and let X be a Banach space containing a sequence of subspaces uniformly complemented and uniformly isomorphic to ℓ_p^n. Then X is not uniformly finitely extensible; hence, it is not automorphic.*

Corollary 4.7: *Let X be a Banach space that contains ℓ_∞^n uniformly and also contains a sequence of finitely dimensional subspaces F_n with $\dim F_n \to \infty$ which are not uniformly isomorphic to the corresponding $\ell_\infty^{dim F_n}$ but are uniformly complemented in X and have uniformly bounded unconditional basic constants. Then X cannot be uniformly finitely extensible; hence it cannot be automorphic.*

Proof. Find a sequence E_n of almost isometric copies of F_n inside suitable $\ell_\infty^{m(n)}$ for large $m(n)$. The spaces E_n cannot be uniformly complemented in $\ell_\infty^{m(n)}$ by [11, p. 298]; and the same in X. ∎

We can give a list of some proposed candidates to be automorphic which actually are not.

Proposition 4.8: *The following spaces are not extensible and, therefore, not automorphic:*

(1) *The spaces L_p and ℓ_p, $1 \leq p < \infty$, $p \neq 2$.*
(2) *The Argyros-Deliyanni hereditarily indecomposable space.*
(3) *Tsirelson's space and its dual.*
(4) *The Gurarij space.*

Part (1) was raised in [13] for ℓ_p with $2 < p < \infty$. Zippin suggested in [26, p. 1729] that the space mentioned in (2) could be automorphic. The Argyros-Deliyanni hereditarily indecomposable space constructed in [2] and the Tsirelson space T [19, Theorem 12] are not uniformly finitely extensible since they contain uniformly complemented ℓ_1^n. The dual T^* contains ℓ_∞^n uniformly and the canonical basis of T^* is unconditional. By Corollary 4.7, T^* is not uniformly finitely extensible.

The Gurarij space G [15] could be expected to be automorphic by the extension property for finite rank operators which it possesses. Nevertheless, G is not even extensible. This follows from the simple observation that complemented subspaces of an extensible space are extensible and from the fact that every separable isometric predual of L_1 is isomorphic to a complemented subspace of G [25]. In particular, $C[0,1]$ is complemented in G and, as we have already seen, it is not extensible. Thus, G is not extensible. Note that G is an $\mathcal{L}_\infty$-space, so *is* uniformly compactly extensible.

5. Further remarks and open problems

About stability properties of automorphic spaces we can only assert that subspaces of an automorphic space do not need to be automorphic.

Proposition 5.1: *A subspace X of c_0 is extensible if and only if it is isomorphic to c_0. In particular, for any sequence of finite dimensional spaces E_n, the vector sum $(\sum_1^\infty E_n)_{c_0}$ is extensible (automorphic) if and only if it is isomorphic to c_0.*

Proof. It is well-known that a subspace X of c_0 is isomorphic to c_0 if and only if it is complemented in c_0 (see [14, Theorem 2.a.3]). Also it is well-known ([14, Proposition 2.a.2]) that X contains a complemented subspace which is isomorphic to c_0. So, $X \sim Y \oplus c_0 \sim Y \oplus c_0 \oplus c_0 \sim X \oplus c_0$, hence, X contains a complemented copy of itself. If X was not complemented in c_0, then $X \sim Y \oplus c_0$ would also contain a uncomplemented copy of X. By Lemma 4.3, X cannot be extensible. ∎

While complemented subspaces of extensible spaces are extensible, we do not know whether complemented subspaces of automorphic spaces are automorphic. The following proposition shows a simple necessary condition for being automorphic.

Proposition 5.2: *Let X be a Banach space isomorphic to its square. If X admits a quotient without a complemented copy of X, then X cannot be automorphic.*

Proof. Let E be a subspace of X such that X/E contains no complemented copy of X. Since $X \sim X \oplus X$, X contains a subspace F isomorphic to E such that X/F contains a complemented copy of X. Of course, one cannot extend the isomorphism between E and F to an automorphism of X. ∎

Let us notice that $C(K)$ and L_p are not extensible for different reasons: $C(K)$ is compactly extensible (see [26, Theorem 4.3]), while L_p is not. Since $\mathcal{L}_\infty$-spaces are compactly extensible (see [10] or [26, Theorem 4.2]), Theorem 4.4 immediately yields that a $\mathcal{L}_\infty$-space X cannot contain sequences of pairwise uniformly isomorphic subspaces (E_n) and (F_n) satisfying (2) in Theorem 4.4 with $X = Y$. We already know that infinite dimensional separable $C(K)$-spaces non-isomorphic to c_0 cannot be automorphic. This suggests

Question 2: Does there exist a non-separable automorphic $C(K)$-space not isomorphic to $c_0(\Gamma)$? Does there exist an automorphic $\mathcal{L}_\infty$-space non-isomorphic to $c_0(\Gamma)$?

Question 3: Let X be a Banach space of finite cotype which is not isomorphic to a Hilbert space. Does X contain sequences of subspaces (E_n) and (F_n), $\dim E_n = \dim F_n$, satisfying (2) in Theorem 4.4 with $X = Y$?

The local approach to automorphic spaces in Section 4 suggests it would be reasonable to study the extensible or automorphic character of spaces of finite dimension, something which only makes sense considering quantitative estimates.

Question 4: Let $1 < p < \infty$, $p \neq 2$. For which λ is the space ℓ_p^n λ-extensible?

It is clear that ℓ_2^n and ℓ_∞^n are 1-extensible.

Question 5: Let $\lambda \geq 1$ and, for every $n \in \mathbb{N}$, let X_n be an n-dimensional λ-extensible space. Is it true that

$$\sup_n \min\{\mathrm{d}(X_n, \ell_2^n), \mathrm{d}(X_n, \ell_\infty^n)\} < \infty \text{ ?}$$

We do not even know whether there is some $\lambda > 1$ and a sequence of n-dimensional $\mathbf{P}_\lambda$-spaces E_n such that $\mathrm{d}(E_n, \ell_\infty^n) \to \infty$ (see [26, p. 1716]).

Definition 5.1 (λ-automorphic space): Let $\lambda \geq 1$. We say that a finite dimensional normed space X is λ-automorphic if for every subspace $E \subset X$ and every into isomorphism $T : E \to X$ there is an automorphism $\widehat{T} : X \to X$ extending T such that $\|\widehat{T}\|\|\widehat{T}^{-1}\| \leq \lambda\|T\|\|T^{-1}\|$.

Obviously, ℓ_2^n is 1-automorphic for every n. But, maybe surprisingly, we do not know the automorphic character of the spaces ℓ_∞^n. What we can say is that ℓ_∞^n cannot be λ-automorphic for λ close to 1. To check this take the unit vector basis (e_i) of ℓ_∞^n, $e = (1, 1, \ldots, 1)$ and set $Te_1 = e$. If $\|\widehat{T}e_2\| > 1/2$ then $\|e_1 + e_2\| = \|e_1 - e_2\| = 1$ and $\|\widehat{T}(e_1 + e_2)\| > 3/2$ or $\|\widehat{T}(e_1 - e_2)\| > 3/2$.

The following variation of Question 5 is related to the automorphic character of finite dimensional spaces and can be considered as a kind of *rotation problem*:

Question 6: For every n, let X_n be a λ-automorphic space with $\dim X_n = n$. Is it true that $\sup_n \mathrm{d}(X, \ell_2^n) < \infty$?

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